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# CAP-WEIGHTED PORTFOLIOS ARE SUB-OPTIMAL PORTFOLIOS[1]

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*In this paper, we show that under a fairly innocuous assumption on price inefficiency, market capitalization weighted portfolios are sub-optimal. If market prices are more volatile than is warranted by changes in firm fundamentals, then cap-weighted portfolios do not capture the full premium commensurate their risk. The sub-optimality arises because cap-weighting tends to overweight stocks whose prices are high relative to their fundamentals and underweight stocks whose prices are low relative to their fundamentals. The size of the cap-weighted portfolio underperformance is increasing in the magnitude of price inefficiency and is roughly equal to the variance of the noise in prices. However, portfolios constructed from weights, which do not depend on prices, do not exhibit the same underperformance observed for cap-weighted portfolios. We illustrate this cap-weighting underperformance empirically by comparing returns from cap-weighted portfolio versus non-cap-weighted portfolios with similar characteristics. We also derive testable implications from our model assumption and find empirical support.*

## 1 Rationale for market capitalization portfolio weights

Before we introduce the logic and mathematics on why cap-weighted portfolios might be sub-optimal, it is important for us to review the merits of cap-weighting. The benefits of a cap-weighting portfolio strategy are numerous. We list the most notable ones below:

1. Cap-weighting is a passive strategy requiring no (little) active management and therefore no active management fee.
2. Cap-weighted portfolios are automatically rebalanced as security prices fluctuate. There is no rebalancing cost associated with executing this strategy except for replacing a constituent security in the portfolio.
3. Cap-weighting assigns the greatest weights to the largest companies. Since market capitalization is highly correlated with liquidity, cap-weighting

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ensures that the portfolio is mostly invested in highly liquid stocks, thus reducing expected portfolio transaction costs.

4. Under a "standard" interpretation of the Capital Asset Pricing Model, a broadbased cap-weighted portfolio (a "market" portfolio) is automatically Sharpe Ratio maximized (or mean-variance optimality).

The benefits listed in (1)–(3) are widely accepted and require no assumptions. However, the mean-variance optimality stated in (4) is obtained only when very specific assumptions hold.[2] Today, more than 1 trillion dollars are invested in passive cap-weighted indexes. In that context, whether passive cap-weight indexing is an optimal portfolio strategy becomes particularly important to the investment community. We show, in the next section, that with very mild price inefficiency in the market, cap-weighting would not be an optimal portfolio construction.

## 2 Why cap-weighted portfolios might be sub-optimal

Under fairly innocuous assumptions on the stock pricing process, we can show that cap-weighting is a sub-optimal portfolio strategy. First, we demonstrate theoretically that the performance of a cap-weighted portfolio is lower than otherwise similar non-cap-weighted portfolios. We strengthen our case further by showing empirical evidence of this underperformance when a cap-weighted portfolio is benchmarked against non-cap-weighted passive portfolios of similar risk characteristics.

The intuition for the cap-weighting underperformance is simple. If stock prices are inefficient in the sense that they do not fully reflect firm fundamentals, then under-priced stocks will have smaller capitalizations than their fair equity value and similarly over-priced stocks will have larger capitalizations than their fair equity value. A cap-weighted portfolio would on average shift additional weights into the over-priced stocks and shift weights away from the under-priced stocks. As long as these pricing errors are not persistent, market prices will collapse toward fair value over time and a cap-weighted portfolio would tend to experience greater price decline than other non-price-weighted portfolios due to its heavier exposure to stocks with positive pricing error.

In the following example, we illustrate the sub-optimality of cap-weighting using a simple binomial one period model. In Section 3, we introduce the mathematics, which make the intuition precise and relate the underperformance of a cap-weighted portfolio to the level of noise (or size of price inefficiency) in stock prices.

### 2.1 *A binomial example of cap-weighting sub-optimality*

Suppose there are only two stocks in the market, A and B, each with one share outstanding. Suppose the fair fundamental values (which investors do not observe) are \$10 per share for each stock. Further, suppose that market prices are noisy, and that there is a 50/50 chance that a stock can be overvalued or undervalued by \$2 (equivalent to assuming a 20% noise in price). Note that the expected "mispricing" in either of the two stocks is *zero* and we cannot know which stock is overvalued or undervalued. For simplicity, we also assume that the two stocks have the same systematic factor exposure (same market beta in the CAPM context), which leads to a 10% return on equity capital.

Observe that the cap-weighted market portfolio has $\left(\frac{12}{12+8}\right) = 60\%$ in the overvalued stock and $\left(\frac{8}{12+8}\right) = 40\%$ in the undervalued stock. However, had prices reflected fundamentals, the portfolio weight would have been $\left(\frac{10}{10+10}\right) = 50\%$ in each. After one period, even assuming that the overvaluation and undervaluation does not dissipate, the cap-weighted portfolio return would be $\left(60\% \cdot \frac{\$10 \cdot 10\%}{\$12} + 40\% \cdot \frac{\$10 \cdot 10\%}{\$8}\right) = 10.00\%$. However, had the "fair-value-weight" been applied, the "fair-value-portfolio" would earn a return of $\left(50\% \cdot \frac{10 \cdot 10\%}{12} + 50\% \cdot \frac{10 \cdot 10\%}{8}\right) = 10.42\%$. The intuition for the cap-weighted portfolio's return drag is clear. The cap-weighted portfolio underperforms, because it puts more weight in the overvalued stock and less weight in the undervalued stocks.

The return drag is clearly related to the over/undervaluation. Suppose in this example the mispricing was \$3 (30%), instead of \$2, the return drag on the cap portfolio relative to the fair-value-weighted portfolio would be 0.99%. At \$4 and \$5 mispricing the return drags are 1.90% and 3.33%, respectively!

Furthermore, suppose the mispricing is transient, meaning that it largely dissipates over the course of the holding period, the return drag on the cap-weighted portfolio is even more substantial. Returning to our original example with \$2 mispricing, if we additionally assume the dissipation of pricing noise, the overpriced stock would revert from \$12 back toward \$10 while the underpriced stock would move from \$8 toward \$10, in addition to the co-movement with the equity market factor(s). Therefore, the cap-weighted portfolio return would be $\left(60\% \cdot \frac{\$10 \cdot 10\% + (\$10 - \$12)}{\$12} + 40\% \cdot \frac{\$10 \cdot 10\% + (\$10 - \$8)}{\$8}\right) = 10\%$ while the fair-value-weighted portfolio would return $\left(50\% \cdot \frac{\$10 \cdot 10\% + (\$10 - \$12)}{\$12} + 50\% \cdot \frac{\$10 \cdot 10\% + (\$10 - \$8)}{\$8}\right) = 14.58\%$. We note that the return drag is significantly increased when pricing noises are not persistent.

We illustrate in this simple example that when prices are noisy, the cap-weighted portfolio underperforms the fair-value-weighted portfolio significantly. In later sections, we show explicitly that when the noise is not persistent the cap-weighted portfolio underperformance is in general equal to twice the variance of the pricing noise. We show in Section 3 a general mathematical derivation of this result and expand on the assumptions and intuitions that lead to cap-weighting underperformance.

## 3 Portfolio behaviors with price inefficiencies

### 3.1 *Expected return in an economy with mispricing risk*

Let $\widehat{P}_{i,t}$ be the observed market price for stock $i$ at time $t$. We make the assumption that the market price is noisy and does not always fully reflect firm fundamentals. That is prices are more volatile than is warranted by changes in firm value. Consequently, a stock is either too expensive or too cheap relative to its fundamental value $P^*_{i,t}$. The assumption on noisy market prices (price inefficiency) is empirically well motivated. Shiller (1981) and LeRoy and Porter (1981) find that stock price volatility is too high relative to subsequent changes in dividends. Brainard *et al.* (1990) among others find that movements in the stock prices are difficult to reconcile with rational changes in beliefs on future cashflows or interest rates. In fact, both Keynes (1964) and Williams (1956) observed early on that events, which have small to no longer-term impacts, tend to contribute to an excessive and absurd degree on price movements.

Based on our assumed form of price inefficiency, we can decompose stock price into two components:

$$\widehat{P}_{i,t} = P^*_{i,t}(1 + \varepsilon_{i,t}) \tag{1}$$

where $P^*_{i,t}$ is the *theoretical* fair price (which is unavailable/unobservable to market participants), which reflects stock $i$'s fundamentals at time $t$, and where $\varepsilon_{i,t}$ is a white noise term, with mean zero and variance $\sigma^2$, capturing the over/undervaluation at time $t$.[3,4] Note that the price noise is assumed i.i.d. and therefore is not persistent. We can allow persistency in the noise; however, it does not qualitatively change our analysis. Also note that the noise recurs every period; that is, the observed market price never collapses back to the fundamental price. It is important to stress here that the price inefficiency we assume here is mild. There is no direct way to take advantage of this inefficiency. Our only knowledge is that some stocks are overpriced while others are underpriced relative to their fair price. However, we do not know which stock is overpriced or underpriced.

The holding period return is

$$1 + \widehat{R}_{i,t+1} = \frac{\widehat{P}_{i,t+1}}{\widehat{P}_{i,t}} = \frac{P^*_{i,t+1}(1 + \varepsilon_{i,t+1})}{P^*_{i,t}(1 + \varepsilon_{i,t})} \quad (2)$$

which can be approximated using second-order Taylor expansion (almost exactly) by[5]

$$1 + \widehat{R}_{i,t+1} = \frac{P^*_{i,t+1}(1 + \varepsilon_{i,t+1})}{P^*_{i,t}}(1 - \varepsilon_{i,t} + \varepsilon^2_{i,t}) \quad (3)$$

Multiplying, rearranging, and dropping terms higher than second order, we have

$$1 + \widehat{R}_{i,t+1} = \frac{P^*_{i,t+1}}{P^*_{i,t}}(1 + \varepsilon_{i,t+1} - \varepsilon_{i,t} + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1}) \quad (4)$$

For ease of notation, we define the true net stock price appreciation as

$$1 + R^*_{i,t+1} = \frac{P^*_{t+1}}{P^*_t} \quad (5)$$

Substituting (5) into (4) gives us

$$1 + \widehat{R}_{i,t+1} = (1 + R^*_{i,t+1})(1 + \varepsilon_{i,t+1} - \varepsilon_{i,t} + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1}) \quad (6)$$

Applying the expectation operator to Eq. (6) and dropping terms with order greater than the variance term $\sigma^2$, the expected return for holding stock $i$ in this economy is[6]

$$E[\widehat{R}_{i,t+1}] = E[R^*_{i,t+1}] + \sigma^2(1 + E[R^*_{i,t+1}]) \quad (7)$$

From (7), we note that the expected return, $E[\widehat{R}_{i,t+1}]$, from holding stock $i$ in our economy with random mispricing is $\sigma^2(1 + E[R^*_{i,t+1}])$ higher than the expected holding period return, $E[R^*_{i,t+1}]$, in an ideal economy without "mispricing." The augmented expected return can be *interpreted* as "premium" offered for holding stocks in an economy with mispricing risk. Intuitively, the higher equity premium offered compensates investors for bearing the mispricing risk, in addition to holding the firm's fundamental economic risk; we note that this is consistent with the intuition that information inefficiency in markets lead to higher cost of capital.[7] Since the paper does not present a general equilibrium model, it is important that we do not overstate our interpretation for the additional premium term in Eq. (7).

We examine, in the following sections, the expected return on a cap-weighted portfolio. Our derivation, which shows that a cap-weighted portfolio would on average produce a negative alpha relative to its fair expected return. Specifically, when the portfolio's fair expected return (given its risk) is $R$, the cap-weighting construction would instead produce an expected return of $R - \sigma^2(1 + R)$—or an alpha of $-\sigma^2(1 + R)$.

### 3.2 *Expected returns for cap-weighted portfolios*

To make our argument formal, we construct a cap-weighted portfolio and examine its expected return characteristic. By the definition of cap-weighting, the weight for stock $i$ in a cap-weighted portfolio (of $N$ stocks) is

$$\widehat{w}_{i,t} = \frac{\widehat{P}_{i,t} \cdot S_i}{\sum_{k=1}^{N} \widehat{P}_{k,t} \cdot S_k} \tag{8}$$

where $S_i$ is the number of shares outstanding for stock $i$, and $\widehat{P}_{i,t}$ is the market price, and where the denominator $\sum_{k=1}^{N} \widehat{P}_{k,t} \cdot S_k$ is the weighted market capitalization of the portfolio and the numerator $\widehat{P}_{i,t} \cdot S_i$ is the market capitalization of stock $i$.

Substituting (1) into (8), we get

$$\widehat{w}_{i,t} = \frac{P^*_{i,t}(1+\varepsilon_{i,t}) \cdot S_i}{\sum_{k=1}^{N} P^*_{k,t}(1+\varepsilon_{k,t}) \cdot S_k} \tag{9}$$

where $P^*_{i,t}$ and $\varepsilon_{i,t}$, again, are the *fair fundamental price* and the *noise* in market price for stock $i$.

Rearranging the denominator in (9), we have

$$\begin{aligned}\widehat{w}_{i,t} &= \frac{P^*_{i,t}(1+\varepsilon_{i,t}) \cdot S_i}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k + \sum_{k=1}^{N} P^*_{k,t} \cdot S_k \cdot \varepsilon_{k,t}} \\ &= \frac{P^*_{i,t} \cdot S_i(1+\varepsilon_{i,t})}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k \left(1 + \frac{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k \cdot \varepsilon_{k,t}}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k}\right)}\end{aligned} \tag{10}$$

which can be approximated by

$$\begin{aligned}\widehat{w}_{i,t} = {} & \frac{P^*_{i,t} \cdot S_i}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k}(1+\varepsilon_{i,t}) \\ & \times \left(1 - \frac{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k \cdot \varepsilon_{k,t}}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k} + \left(\frac{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k \cdot \varepsilon_{k,t}}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k}\right)^2\right)\end{aligned} \tag{11}$$

To simplify the notation in Eq. (11), let

$$\bar{\varepsilon}_t = \frac{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k \cdot \varepsilon_{k,t}}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k} \tag{12}$$

Substituting (12) into (11) and rearranging, we have

$$\widehat{w}_{i,t} = \frac{P^*_{i,t} \cdot S_i}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k}(1+\varepsilon_{i,t})(1 - \bar{\varepsilon}_{i,t} + \bar{\varepsilon}^2_{i,t}) \tag{13}$$

Had we been able to observe the fair price for firms, we would know the true capitalization weights:

$$w^*_{i,t} = \frac{P^*_{i,t} \cdot S_i}{\sum_{k=1}^{N} P^*_{k,t} \cdot S_k} \tag{14}$$

Substituting (14) into (13), and expanding and dropping higher-order terms, we have

$$\widehat{w}_{i,t} = w^*_{i,t}(1 + \varepsilon_{i,t} - \bar{\varepsilon}_{i,t} + \bar{\varepsilon}^2_{i,t} - \varepsilon_{i,t}\bar{\varepsilon}_{i,t}) \tag{15}$$

By definition, the return of a portfolio $P$ is the weighted average return of the individual stocks in the portfolio:

$$\widehat{R}_{P,t+1} = \sum_{i=1}^{N} \widehat{R}_{i,t+1}\widehat{w}_{i,t} \tag{16}$$

Substituting Eqs. (6) and (15) into Eq. (16), we have

$$\begin{aligned}\widehat{R}_{P,t+1} = {} & \sum_{i=1}^{N} w^*_{i,t}(1 + \varepsilon_{i,t} - \bar{\varepsilon}_{i,t} + \bar{\varepsilon}^2_{i,t} - \varepsilon_{i,t}\bar{\varepsilon}_{i,t}) \\ & \times \{(1 + R^*_{i,t+1})(1 + \varepsilon_{i,t+1} - \varepsilon_{i,t} \\ & + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1}) - 1\}\end{aligned} \tag{17}$$

Expanding Eq. (17), and then taking expectation (noting that the white noise term is uncorrelated across assets and across time) and dropping terms with orders greater than $\sigma^2$, we have

$$E[\widehat{R}_{P,t+1}] = (1-\sigma^2) \cdot \sum_{i=1}^{N} E[R^*_{i,t+1} w^*_{i,t}] - \sigma^2 \tag{18}$$

We denote $E[R^*_{P,t}] = \sum_{i=1}^{N} E[R^*_{i,t+1} w^*_{i,t}]$ as the expected return on the portfolio that is attributable to the firm fundamental risk associated with the stocks in the portfolio; then, (18) becomes

$$E[\widehat{R}_{P,t+1}] = E[R^*_{P,t+1}] - \sigma^2(1 + E[R^*_{P,t+1}]) \tag{19}$$

Recalling Eq. (7), we know that securities in our economy with mispricing risk earn expected returns that are higher than what is warranted by their *firm fundamental risk*:

$$E[\widehat{R}_{i,t}] = E[R^*_{i,t}] + \sigma^2(1 + E[R^*_{i,t}]) \tag{20}$$

However, from Eq. (18), we see that the cap-weighted portfolio $P$ earns an expected return only equal to $E[R^*_{P,t+1}] - \sigma^2(1 + E[R^*_{P,t+1}])$, which is $\sigma^2(1 + E[R^*_{i,t}])$ below what its portfolio of *firm fundamental risks* would commend. We explore the source of this *return drag* on cap-weighted portfolios in a later section. For now, we note that this return drag increases with the size of the price inefficiency ($\sigma^2$) in the equity market. In the next section, we show that portfolios, which are not cap-weighted, do not exhibit return drags.

### 3.3 Expected returns for non-cap-weighted portfolios

In this section, we show that alternative-size-weighting schemes, which do not depend on market capitalization, do not suffer from the return drag illustrated in the previous section. Suppose we use some other measure of firm size instead of market capitalization to create portfolio weights. For simplicity, let us suppose we do an admirable job at creating the weights so that

$$\widehat{w}_{i,t} = w^*_{i,t}(1 + \nu_{i,t}) \tag{21}$$

where $\nu_{i,t}$ is a mean zero white noise uncorrelated with other random variables. This is to say that the selected portfolio weights may deviate significantly and across the board from the "true-value-weight," but these mistakes in assigning weights are not related to other variables, such as market prices or firm capitalization.

The return for a portfolio $P$ is

$$\widehat{R}_{P,t+1} = \sum_{i=1}^{N} \widehat{R}_{i,t+1}\widehat{w}_{i,t} \tag{22}$$

Substituting (6) and (21) into (22), we have

$$\begin{aligned}\widehat{R}_{P,t+1} = \sum_{i=1}^{N} w^*_{i,t}(1+\nu_{i,t})\{&(1 + R^*_{i,t+1}) \\ &\times (1 + \varepsilon_{i,t+1} - \varepsilon_{i,t} + \varepsilon^2_{i,t} \\ &- \varepsilon_{i,t}\varepsilon_{i,t+1}) - 1\}\end{aligned} \tag{23}$$

Since the portfolio weights $\widehat{w}_{i,t} = w^*_{i,t}(1 + \nu_{i,t})$ are assumed to not depend on market capitalizations or market prices $\widehat{p}_{i,t}$, we have $E[\nu_{i,t}\varepsilon_{i,t}] = 0$. Applying the expectation operator on Eq. (23), we have

$$\begin{aligned}E[\widehat{R}_{P,t+1}] = &\sum_{i=1}^{N} E[w^*_{i,t}R^*_{i,t+1}] \\ &+ \sigma^2\left(1 + \sum_{i=1}^{N} E[w^*_{i,t}R^*_{i,t+1}]\right)\end{aligned} \tag{24}$$

Recall that we define $E[R^*_{P,t}] = \sum_{i=1}^{N} E[R^*_{i,t+1}w^*_{i,t}]$. Equation (24) is then rewritten as

$$E[\widehat{R}_{P,t+1}] = E[R^*_{i,t+1}] + \sigma^2(1 + E[R^*_{i,t+1}]) \tag{25}$$

This alternative-size-weighted portfolio earns an expected return that is $\sigma^2(1 + E[R^*_{i,t+1}])$ above what its portfolio of *firm fundamental risks* would commend. Again, we interpret the additional return as compensation to investors for mispricing risk in the economy. Comparing the expected return for a cap-weighted portfolio against the expected return for a non-cap-weighted portfolio, we find that a non-cap-weighted portfolio with similar characteristics (in terms of the underlying firm fundamental risks) would outperform by $2\sigma^2(1 + E[R^*_{i,t+1}])$.

## 4 Economic significance of the underperformance from cap-weighting

From Section 3, we show that when market prices are noisy, a cap-weighted portfolio would underperform an alternative-size-weighted portfolio by $2\sigma^2(1 + E[R^*_{i,t+1}])$. We examine the economic significance of this result. The average stock return volatility is roughly 40% per annum. We assume that 5% of the total stock return variance ($0.05 * 0.4 * 0.4 = 0.008$) is attributable to noise trading—that is, 5% of the total price movements are not related to changes in the firm's fundamentals. The stock market returns for the last half-century have averaged about 11% per annum. This suggests that the cap-weighted portfolio would underperform by $2 * 0.008 * 1.11 = 1.78\%$, or 178 basis points per annum on average relative to an alternative-size-weighted portfolio. This suggests that return drag can be very significant even when only a small amount of price noise is present.

However, it is important to examine whether the cost of implementing an alternative-size-weighted portfolio would be higher than 178 bps per annum. The most notable objection to a non-market capitalization-based portfolio strategy is the transactions cost associated with rebalancing. In periods of large price movements, an alternative-size-weighted portfolio will drift significantly relative to the original policy weights. Rebalancing back to the desired policy weights could incur additional trading costs, which are not present in a cap-weighted portfolio strategy. At a 2% round trip total trading cost, the alternative-size-weighted portfolio would have to turnover 89% more than the cap-weighted portfolio to negate its performance advantage.

## 5 Empirical predictions

### *5.1 Negative return autocorrelation*

Crucial to our derivation, which shows that cap-weighting is sub-optimal relative to alternative-size-weighting, is the assumption that stock returns are more noisy than warranted by the changes in the underlying firm fundamentals. We have already discussed the empirical literature on excess price volatility in the stock market in Section 3 to motivate this assumption. In this section, we derive a simple testable relationship to further verify this assumption.

The noisy stock price assumption from Section 3 suggests that stock returns are serially negatively correlated. This can be shown easily by examining the covariance between returns at time $t$ and $t + 1$. Recalling from Eq. (6), stock returns at time $t$ and $t + 1$ are described by

$$\begin{aligned} 1 + \widehat{R}_{i,t} &= (1 + R^*_{i,t})(1 + \varepsilon_{i,t} - \varepsilon_{i,t-1} \\ &\quad + \varepsilon^2_{i,t-1} - \varepsilon_{i,t-1}\varepsilon_{i,t}), \qquad (26) \\ 1 + \widehat{R}_{i,t+1} &= (1 + R^*_{i,t+1})(1 + \varepsilon_{i,t+1} - \varepsilon_{i,t} \\ &\quad + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1}) \qquad (27) \end{aligned}$$

Using the linearity of the covariance operator, we have

$$\begin{aligned} &\mathrm{cov}(\widehat{R}_t, \widehat{R}_{t+1}) \\ &\quad = \mathrm{cov}(\varepsilon_{i,t} - \varepsilon_{i,t-1} + \varepsilon^2_{i,t-1} - \varepsilon_{i,t-1}\varepsilon_{i,t}, \varepsilon_{i,t+1} \\ &\qquad - \varepsilon_{i,t} + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1}) \\ &\qquad + \mathrm{cov}(R^*_{i,t}(\varepsilon_{i,t} - \varepsilon_{i,t-1} \\ &\qquad + \varepsilon^2_{i,t-1} - \varepsilon_{i,t-1}\varepsilon_{i,t}), R^*_{i,t+1}(\varepsilon_{i,t+1} \\ &\qquad - \varepsilon_{i,t} + \varepsilon^2_{i,t} - \varepsilon_{i,t}\varepsilon_{i,t+1})) \qquad (28) \end{aligned}$$

Since the white noise terms ($\varepsilon_{i,t}$) and true stock returns ($R^*_{i,t}$) are independent random variables,

(28) reduces to

$$\begin{aligned}\operatorname{cov}(\widehat{R}_t, \widehat{R}_{t+1}) &= \operatorname{cov}(\varepsilon_{i,t} - \varepsilon_{i,t-1} + \varepsilon_{i,t-1}^2 - \varepsilon_{i,t-1}\varepsilon_{i,t}, \varepsilon_{i,t+1} \\ &\quad - \varepsilon_{i,t} + \varepsilon_{i,t}^2 - \varepsilon_{i,t}\varepsilon_{i,t+1})\end{aligned} \tag{29}$$

Dropping higher-order terms, we have

$$\begin{aligned}\operatorname{cov}(\widehat{R}_t, \widehat{R}_{t+1}) &\approx \operatorname{cov}(\varepsilon_{i,t} - \varepsilon_{i,t-1}, \varepsilon_{i,t+1} - \varepsilon_{i,t}) \\ &= -\sigma^2\end{aligned} \tag{30}$$

It follows directly from Eq. (30) that our model predicts a negative autocorrelation in stock returns. This prediction is consistent with the existing empirical literature. There are strong direct and indirect empirical evidences on negative autocorrelation for cross-sectional stock returns at longer horizons. Poterba and Summers (1988), Fama and French (1988), Lo and MacKinlay (1988, 1990), Jegadeesh (1990), and Kim *et al.* (1991) offer direct evidences on negative stock return autocorrelation at various horizons from monthly returns to 5-year returns. De Bondt and Thaler (1985, 1987), using stocks, which have outperformed, and stocks, which have underperformed, find significant price reversal. Lakonishok *et al.* (1994) also offer indirect evidence on price reversion by identifying successful contrarian strategies.

The empirical literature is consistent with our conjecture of excess price volatility and the derived prediction on negative stock return autocorrelation. We show next that the key prediction from this paper, which predicts that a cap-weighted portfolio would underperform a non-cap-weighted portfolio with similar characteristics, is also supported by empirical data.

### *5.2 Non-cap-weighted portfolios versus market-cap-weighted portfolios*

We establish the validity of the model assumption in this section. Now we verify that the key model prediction—that cap-weighted portfolios will underperform their non-cap-weighted equivalents—is supported by data. We also illustrate the economic significance of this cap-weighting underperformance, which should be of particular importance to the practitioner community involved with passive index investment. We report results from Arnott *et al.* (2005), which compares the performance of a cap-weighted portfolio constructed from 1000 largest capitalization stocks with three non-cap-weighted portfolios (using alternative measures of firm size—book, income, and sales) constructed from 1000 largest stocks as measured by their respective metrics.

Tables 1 and 2 paraphrase the relevant results from Arnott *et al.* (2005). The portfolio construction methodology and the return characteristics are described in detail in the original paper. Table 1 shows that the three alternative-size-weighted portfolios outperformed the cap-weighted portfolio by 2.15% per annum on average over the last 42 years. Table 2 shows an average outperformance of 2.32% per annum when we adjust for CAPM beta risk. These excess returns over the cap-weighted portfolios are statistically significant in addition to being economically significant. Examining volatilities, correlations, and betas, we find the alternative-size-weighted portfolios to be fairly similar in their risk characteristics with the cap-weighted portfolio. This gives confidence that the observed excess returns are not driven by additional risks in the portfolios.

It would be presumptuous for us to assume that the cap-weighted portfolio underperformance reported in Arnott *et al.* (2005) is due entirely to noisy stock prices. Certainly, as Arnott *et al.* suggest in their paper there could be additional hidden risk factors, which these selected non-cap-weighted portfolios may be exposed to.

**Table 1** Cap-weighted portfolio versus alternative-size-weighted portfolios (1962–2003).

| | Geometric return | Volatility | Sharpe ratio | Excess return vs. CAP 1000 | Tracking error vs. CAP 1000 | $t$-Stat for excess return |
|---|---|---|---|---|---|---|
| CAP 1000 | 10.30% | 15.4% | 0.288 | | | |
| BOOK 1000 | 12.02% | 15.0% | 0.409 | 1.72% | 3.54% | 3.16 |
| INCOME 1000 | 12.52% | 15.1% | 0.441 | 2.22% | 3.94% | 3.64 |
| SALES 1000 | 12.80% | 15.9% | 0.434 | 2.50% | 4.93% | 3.28 |
| AVERAGE (BK, INC, Sales) | 12.45% | 15.3% | 0.428 | 2.15% | 4.14% | 3.36 |

**Table 2** Cap-weighted portfolio versus alternative-size-weighted portfolios in CAPM space.

| | Correlation with CAP 1000 | CAPM beta (wrt CAP 1000) | CAPM alpha (wrt CAP 1000) | $t$-Stat for CAPM alpha |
|---|---|---|---|---|
| CAP 1000 | — | — | — | — |
| BOOK 1000 | 97% | 0.95 | 1.94% | 3.63 |
| INCOME 1000 | 97% | 0.95 | 2.45% | 4.12 |
| SALES 1000 | 95% | 0.99 | 2.56% | 3.37 |
| AVERAGE (BK, INC, Sales) | 96% | 0.96 | 2.32% | 3.70 |

## 6 Conclusion

The traditional capitalization-weighting scheme is likely to be sub-optimal if prices are noisy and do not fully reflect firm fundamentals. We provide detailed mathematical proof for this claim and show that the cost of sub-optimal cap-weighting is equal to the square of the noise in the stock prices. Non-cap-weighted portfolios constructions do not suffer from this natural negative alpha associated with cap-weighting. We demonstrate this natural negative alpha empirically to support our claim.

## Notes

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2 Essentially, we need the CAPM assumptions to hold to guarantee that the market clearing portfolio, which by definition is capitalization weighted, is optimal.

3 Note that the stock price $\widehat{P}_{i,t}$ is still an unbiased estimator of the fundamental price $P^*_{i,t}$. There is no systematic way to exploit the price inefficiency we assumed here.

4 Note that $P^*_{i,t}$ is time varying rather than constant across time.

5 In continuous time, this Taylor expansion is, in fact, exact.

6 Suppose that the standard deviation of the pricing noise is 11%; that is, of the 40% in the average stock return volatility, 11% in that volatility is noise. For terms with order higher than $\sigma^2$, the numerical values must be less than $(0.1)^3$, or $< 0.1\%$. Therefore, ignoring these higher-order terms introduces errors in the approximation by less than 10 bps. We will evaluate the impact of this error term more specifically later.

[7] We do not derive a general equilibrium model, which would be outside of the scope of this paper, to make this point exact. The focus of the paper is not on rationalizing why information inefficiency could persist in the equity market. It is primarily focused on the characteristics of cap-weighted portfolios given that there is informational inefficiency and where the price noise is recurring but not infinitely persistent.

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